Last time:

L/K finite ext. of complete, discretely valued fields

$\Rightarrow \exists\, L_0 \subseteq L$ max'l unramified subext. uniquely det. by

- L_0/K is unramified, i.e. k_{L_0} is separable over $k = \mathcal{O}_K/m_K$

  & $k_{L_0} = k_{L,sep}$ = max'l separable subext. of k_L over k

Moreover, L/L_0 is totally ramified (in part $\mathcal{O}_L = \mathcal{O}_{L_0}[\pi_L]$ & the min. poly of π_L is Eisenstein)

If L/K Galois $\Rightarrow$ L_0/K is Galois

Application: Fix a prime p, fix $K/\mathbb{Q}_p$ finite, $\bar{K}$ alg. cl. of K, and $n \geq 1$

$\Rightarrow$ $\{K \subseteq L \subseteq \bar{K},\ [L:K] = n\}$ is finite.

(In contrast to the case of number fields!)

Indeed, $\{K \subseteq L \subseteq \bar{K},\ [L:K] \leq n \ \&\ L/K \text{ unramified}\}$ is finite as $k = \mathcal{O}_K/m_K$ is finite

($\Leftarrow$ $\{k \subseteq l \subseteq \bar{k} = \mathcal{O}_{\bar{K}}/m_{\bar{K}},\ [l:k] \leq n\}$ is finite)

$\Rightarrow$ STP: $\{K \subseteq L \subseteq \bar{K},\ [L:K] = n \ \&\ L/K \text{ totally ramified}\}$ is finite

Applied to K'/K unramified of deg $\leq n$

Consider $M := \{ f \in \mathcal{O}_K[x] \mid f$ monic, Eisenstein of degree $n \}$

$$f = \sum_{i=0}^{n} a_i x^i \longmapsto \left(\frac{a_1}{\pi_K}, \dots, \frac{a_{n-1}}{\pi_K}, \frac{a_0}{\pi_K}\right) \in \mathcal{O}_K^{n-1} \times \mathcal{O}_K^{\times}$$

$M \simeq \mathcal{O}_K^{n-1} \times \mathcal{O}_K^{\times}$ as top. spaces, after fixing a unif. π_K in K

Given $f \in M \Rightarrow \exists$ open nbhd U_f of f, s.t.

$$K[x]/(f(x)) \simeq K[y]/(g(y)) \quad \forall g \in U_f$$

(Thm from last time)

Note:

⚠ M is compact!

$\Rightarrow \quad M = \bigcup_{i=1}^{m} U_{f_i}$ for some $f_1, \dots, f_m \in M$

$\Rightarrow$ Each L/K, $[L:K] = n$, L/K tot. ramified is isom. to $K[x]/(f_i(x))$ for some $i = 1, \dots, m$

Galois extension of complete disc. valued fields

K compl. disc. valued, L/K finite, Galois

Assume k_L separable over k
(automatic if k perfect, e.g. k finite)

In part, $\underset{\text{max'l unram. subext.}}{\underbrace{L_0}} \subseteq L$ Galois

$\Rightarrow k_L/k$ Galois & have short exact sequence

$$1 \to I := I_{L/K} \to Gal(L/K) \to Gal(k_L/k) \to 1$$

$$I = \{\sigma \in Gal(L/K) \mid \sigma(x) \equiv x \bmod m_L \ \forall x \in \mathcal{O}_L\}, \qquad Gal(k_L/k) \cong Gal(L_0/K)$$

"inertia subgroup", $\# I = e(L|K)$

Note: $G := \mathrm{Gal}(L/K)$ acts on $\mathcal{O}_L/\mathfrak{m}_L^i$, $\forall i \geq 0$

Def: $G_i := \ker\left(G \to \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_L/\mathfrak{m}_L^{i+1})\right), i \geq 0$

$\underset{G}{\rotatebox{90}{$\geq$}}$ "higher ramification subgroups"

$$\Rightarrow \quad \begin{array}{c} G_0 \\ \cup \\ G_1 \\ \cup \\ \vdots \end{array} = I_{L/K} \qquad \& \bigcap_{i \geq 0} G_i = \{1\}$$

$$\left(\mathcal{O}_L \simeq \varprojlim_i \mathcal{O}_L/\mathfrak{m}_L^{i+1}\right)$$

Clear: each $G_i \subseteq G$ is normal

Aim: G_i/G_{i+1} is abelian $\forall i \geq 0$

$(\Rightarrow)$ $I_{L/K} = G_0$ is solvable.

If $K/\mathbb{Q}_p$ finite $\Rightarrow k_{L/k}$ is abelian

furthermore

$\Rightarrow \mathrm{Gal}(L/K)$ is solvable

$\Rightarrow L$ is an iterated Kummer extension of K, i.e. it is obtained by iteratively adjoining m-th roots of elements)

Assume L/K is totally ramified (otherwise replace K by L_0)

$\Rightarrow G = I = G_0$

Fix $\pi_L \in \mathcal{O}_L$ uniformizer.

Claim: $G_i = \{ \sigma \in G \mid \frac{\sigma(\pi_L)}{\pi_L} \in U_L^i \}, i \geq 0$

(here $U_L^i = \{ u \in \mathcal{O}_L^\times \mid u \equiv 1 \bmod m_L^i \}$)

Proof: „$\subseteq$" $\sigma \in G_i$

$$\Rightarrow \sigma(\pi_L) \equiv \pi_L \mod (\pi_L^{i+1})$$

$$\underset{\substack{\text{divide by} \\ \pi_L}}{\Rightarrow} \frac{\sigma(\pi_L)}{\pi_L} \equiv 1 \mod (\pi_L)^i = m_L^i$$

„$\supseteq$" Let $x \in \mathcal{O}_L \Rightarrow x = \sum_{j=0}^{\infty} a_j \cdot \pi_L^j$

with $a_j \in \mathcal{O}_K$ (as $k_L = k$)

$$\Rightarrow \quad \sigma(x) = \sum_{j=0}^{\infty} a_j \, \sigma(\pi_L)^j$$

$$\equiv \sum_{j=0}^{\infty} a_j \, \pi_L{}^j \mod (\pi_L)^{i+1}$$

$\square$

Consider, only a map of sets!

$$\tilde{\Theta}_i : G_i \longrightarrow U_L^i \, , \; \sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L}$$

and

$$\Theta_i : G_i \longrightarrow U_L^i / U_L^{i+1} \, , \; \sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L} \mod U_L^{i+1}$$

Then Θ_i is a group hom. with kernel G_{i+1}

Let $\sigma, \tau \in G_i$

$$\Rightarrow \frac{\sigma\tau(\pi_L)}{\pi_L} = \frac{\sigma(\tau(\pi_L))}{\tau(\pi_L)} \cdot \frac{\tau(\pi_L)}{\pi_L}$$

$$\overset{?}{\equiv} \frac{\sigma(\pi_L)}{\pi_L} \cdot \frac{\tau(\pi_L)}{\pi_L} \mod U_L^{i+1}$$

To see ?:

Note $\sigma(\pi_L) = u \cdot \pi_L$ with $u \in \mathcal{O}_L^\times$

because $\sigma(\pi_L), \pi_L$ are uniformizers

$$\Rightarrow \frac{\theta(\sigma(\pi_L))}{\sigma(\pi_L)} = \frac{\sigma(u)\cdot\sigma(\pi_L)}{\underbrace{u}_{\in U_L^1} \cdot \pi_L}$$

$$\Rightarrow \frac{\sigma(\sigma(\pi_L))}{\sigma(\pi_L)} \equiv \frac{\sigma(\pi_L)}{\pi_L} \mod U_L^1$$

$\Rightarrow$? holds

In particular,

abelian!

$$\Theta_i : G_i / G_{i+1} \hookrightarrow U_L^i / U_L^{i+1} \cong \begin{cases} k_L^\times & \text{if } i=0 \\ k_L & \text{if } i \geq 1 \end{cases}$$

use that $\mathcal{O}_L = \varprojlim \mathcal{O}_L / m_L^i$

Corollary: 1) If char $k_L = 0$

$\Rightarrow$ $G_1 = \{1\}$ and G_0 is finite cyclic

Actually, in this case

$$K \simeq k((\pi_K)) \quad \text{and} \quad \overline{K} = \bigcup_{m \geq 1} k((\sqrt[m]{\pi_K}))$$

2) If $\operatorname{char} k_L = p > 0$

$\Rightarrow$ G_1 is a finite p-group, "wild inertia"

G_0/G_1 is finite cyclic of order prime to p

Def: K compl. disc. valued field, L/K finite Galois s.t. k_L sep. over k (prob. not nec.)

Then L/K is tamely ramified if one of the foll. equiv. cond. holds:

1) $p := \operatorname{char} k$ does not divide $e(L/K)$

2) $G_1 = \{1\}$

E.g.: If char $k = 0$ each finite ext. is tamely ramified.

Exercise: $K/\mathbb{Q}_p$ finite, $\overline{K}$ alg. cl. of K

$$\Rightarrow \quad \underbrace{K^{tr}}_{\text{max'l tamely ramified ext. of } K} = \bigcup_{\substack{L \subseteq \overline{K} \\ L/K \text{ tamely ramified}}} L \quad \subseteq \overline{K}$$

Show $K^{tr} = \bigcup_{\substack{m \geq 0 \\ (m,p)=1}} K^{un} \cdot K(\sqrt[m]{\pi_K})$

for each uniformizer π_K of K

In part, $\mathrm{Gal}(K^{tr}/K) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$

Wild ramification is a bit insane:

Let k be a field, $k = \overline{k}$

$$\mathbb{P}^1_k = \operatorname{Spec} k[x] \cup \operatorname{Spec} k[x^{-1}]$$
$$\operatorname{Spec} k[x, x^{-1}]$$

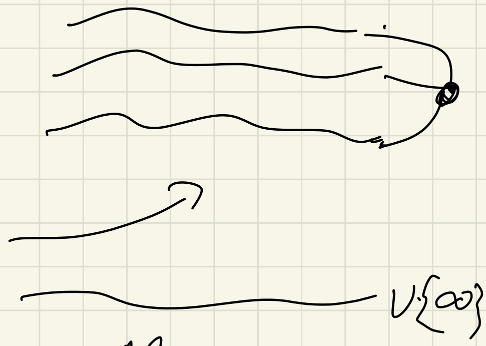
$$\cup \qquad\qquad \psi$$

$$\mathbb{A}^1_k = \operatorname{Spec} k[x] \qquad V(x^{-1}) = \{\infty\}$$

$\{$Finite ext. $L/k(x)$, int. cl. $\mathcal{O}_{L,1}$ of $k[x]$ in L is unramified, & int. cl. $\mathcal{O}_{L,2}$ of $k[x^{-1}]$ in L is tamely ramified at $\infty\}$

$= \{ k(x) \}$

norm of $\mathbb{P}^1_k$ in L



Reason: If char $k = 0$, then $\mathbb{A}^1_k$ is simply connected

(e.g. $k = \mathbb{C} \Rightarrow \mathbb{A}^1_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}$ ← plane is simply connected)

If char $k > 0$, then for each finite simple, non-abelian group H, there exist $L/k(x)$ Galois with Galois group H

& $\mathcal{O}_{L,1}/k[x]$ unramified

(necessarily $\mathcal{O}_{L,2}/k[x^{-1}]$ is wildly ramified at ∞)

$\Rightarrow$ $\mathbb{A}^1_k$ highly non-simply connected

Keyword: Abyankhar's conjecture

$$\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \simeq \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$$

$$\mathrm{Gal}(\mathbb{Q}_p^{ab} \cap \mathbb{Q}_p^{tr}/\mathbb{Q}_p) \simeq \hat{\mathbb{Z}} \times \mathbb{F}_p^\times$$

as $\mathbb{Q}_p^{ab} = \bigcup_{m \geq 1} \mathbb{Q}_p(\mu_m)$ At least if $p \neq 2$

$$\mathbb{Q}_p^{tr} = \bigcup_{\substack{m,n \geq 1 \\ (p,m)=1}} \mathbb{Q}_p(\mu_m, \sqrt[n]{p})$$

and
$$\mathbb{Q}_p^{ab} \cap \mathbb{Q}_p^{tr} = \bigcup_{m \geq 1, (m,p)=1} \mathbb{Q}_p(\mu_m, \mu_p) = \bigcup_{\substack{m \geq 1 \\ (m,p)=1}} \mathbb{Q}_p(\mu_m, \sqrt[p-1]{p})$$

Consider $K^{cyc} := \mathbb{Q}_p(\mu_{p^\infty})$ Assume for simpl. $p \neq 2$

$\Rightarrow$ K^{cyc} is Galois over $\mathbb{Q}_p$ with Galois group

$\mathbb{Z}_p^\times$

and its maximally tamely ramified

subextension is $\mathbb{Q}_p(\mu_p)$

(note $[\mathbb{Q}_p(\mu_p) : \mathbb{Q}_p] = p - 1$

As $\mu_{p-1} \subseteq \mathbb{Q}_p \Rightarrow \mathbb{Q}_p(\mu_p) = \mathbb{Q}_p(\sqrt[p-1]{p})$)

Lubin-Tate

theory: $K/\mathbb{Q}_p$ finite

$\Rightarrow$ Want to prove: $\mathrm{Gal}(K^{ab}/K) \cong \hat{\mathbb{Z}} \times \mathcal{O}_K^\times$

(LCFT)

A. Mihatsch: Gross-Zagier formula

(global arithmetic)

A. Ivanov: Étale cohomology (needs alg. geometry)